Batched Routines in Preconditioning

The Future of Incomplete Factorization Preconditioners

Hartwig Anzt, Thomas Huckle, Jack Dongarra

Workshop on Batched, Reproducible, and Reduced Precision BLAS
http://www.netlib.org/utk/people/JackDongarra/WEB-PAGES/Batched-BLAS-2016/
Innovative Computing Laboratory
University of Tennessee
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INNOVATIVE COMPUTING LABORATORY

THE UNIVERSITY OF TENNESSEE KNOXVILLE

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Many other choices possible for incomplete factorizations!

*Saad: **"Iterative Methods for Sparse Linear Systems (2nd Edition)".** *SIAM, 2003.*

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    - **exact factorization + exact solve = Sparse direct solver**

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- Inherently sequential, level scheduling often provides little parallelism.
- Over-engineering for incomplete factorization preconditioners with $S \subsetneq \mathbb{R}^{n\times n}$?

Preconditioner benefit limited by ILU quality

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Use similar idea like in the factorization step: Approximate on some sparsity pattern!

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➡ Jacobi Iteration $y = D_L^{-1} z + \left(I - D_L^{-1} L\right) y$

-- More Krylov solver iterations may be needed

\+ Faster triangular solves (few SpMVs) can make solution process faster[1,2]

[1]Chow and Patel. **"Fine-grained Parallel Incomplete LU Factorization".** *In: SIAM J. on Sci. Comp. (2015).*
[2]Anzt, Chow, and Dongarra. **"Iterative Sparse Triangular Solves for Preconditioning".** *In: LNCS. 2015.*

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➡ Block Jacobi Iteration

-- Need inverse of diagonal blocks

\+ Good for problems with inherent block-structure (e.g. with FEM origin)[3]

[3]Chow and Scott. **"On the use of iterative methods and blocking for solving sparse triangular systems in incomplete factorization preconditioning".** *Rutherford-Appleton Technical Report RAL-P-2016-006, 2016.*

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$$\approx 0 \text{ for } (M_L^{-1} L - I) \approx 0$$

$$\Leftrightarrow (M_L^{-1} L - M_L^{-1} M_L) \approx 0$$

$$\Leftrightarrow (M_L^{-1} (L - M_L)) \approx 0$$

$$\Leftrightarrow (M_L^{-1} (\underbrace{L M_L^{-1}} M_L - M_L)) \approx 0$$

$$L M_L^{-1} \approx I \text{ i.e. } (L M_L^{-1} = I)_{S^*} \text{ for some } S^*$$

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≈ 0 for $(LM_L^{-1} = I)_{S^*}$ with, e.g. $S^* = spy(A)$

SpAI aproach: choose M_L^{-1} as sparse approximate inverse for L on pattern S^*

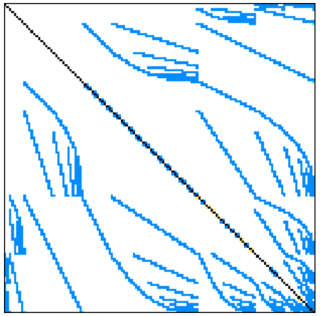
-- Need generation of SpAI matrix

+ One single SpMV for SpAI application

+ Flexibility in choice of the SpAI pattern (CA-Krylov methods...)

[4]Huckle, Anzt, Dongarra **"Parallel Preconditioning"**. *In: SIAM PP 2016*.

Example: ILU - BiCGSTAB



Test matrix: parabolic_fem (n=525,825 nnz=3,674,625)
Use RCM-reordering

BiCGSTAB Preconditioning	Iterations	Prec. setup time	Solver runtime	Total runtime
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NVIDIA K40 GPU, CUDA/cuSPARSE v7.0

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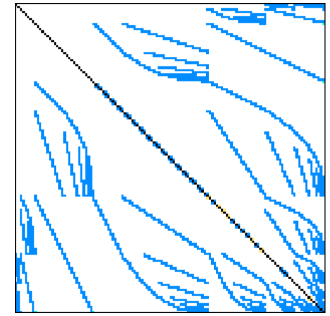

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~ 3x

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Generating SpAI for triangular factor

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Use M for M_L^{-1}:

```
[L U] = ILU(A);
M = spones(L);                  % also possible L^2, L^3 ...
n=length(L);
for i=1:n
   J = find(M(:,i));            % determine nonzero entries
   generate L(J,J);             % small triangular system
   solve L(J,J) M(J,i) = e_i(J);  % one trsv per column of M
   insert M(J,i) into M;        % insert data into SpAI matrix M
end
```

Same strategy for SpAI matrix approximating upper triangular factor.

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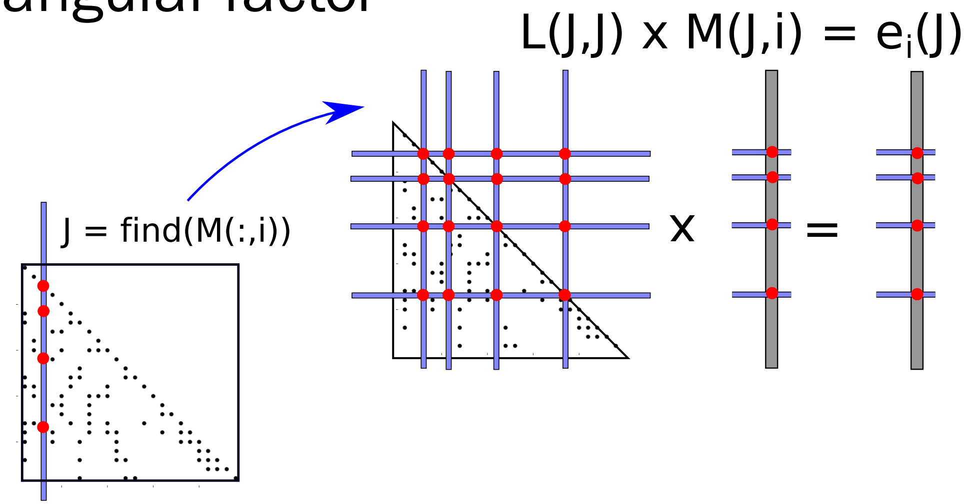

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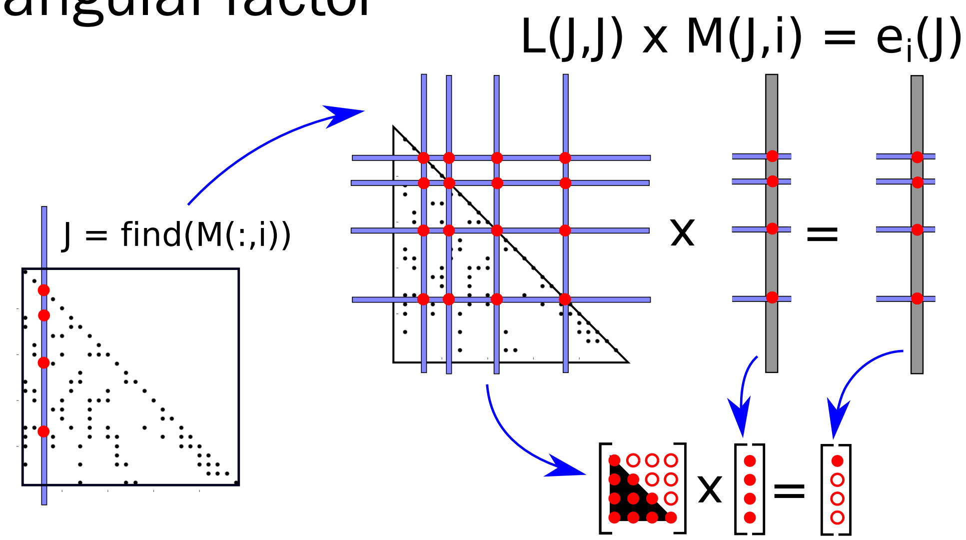

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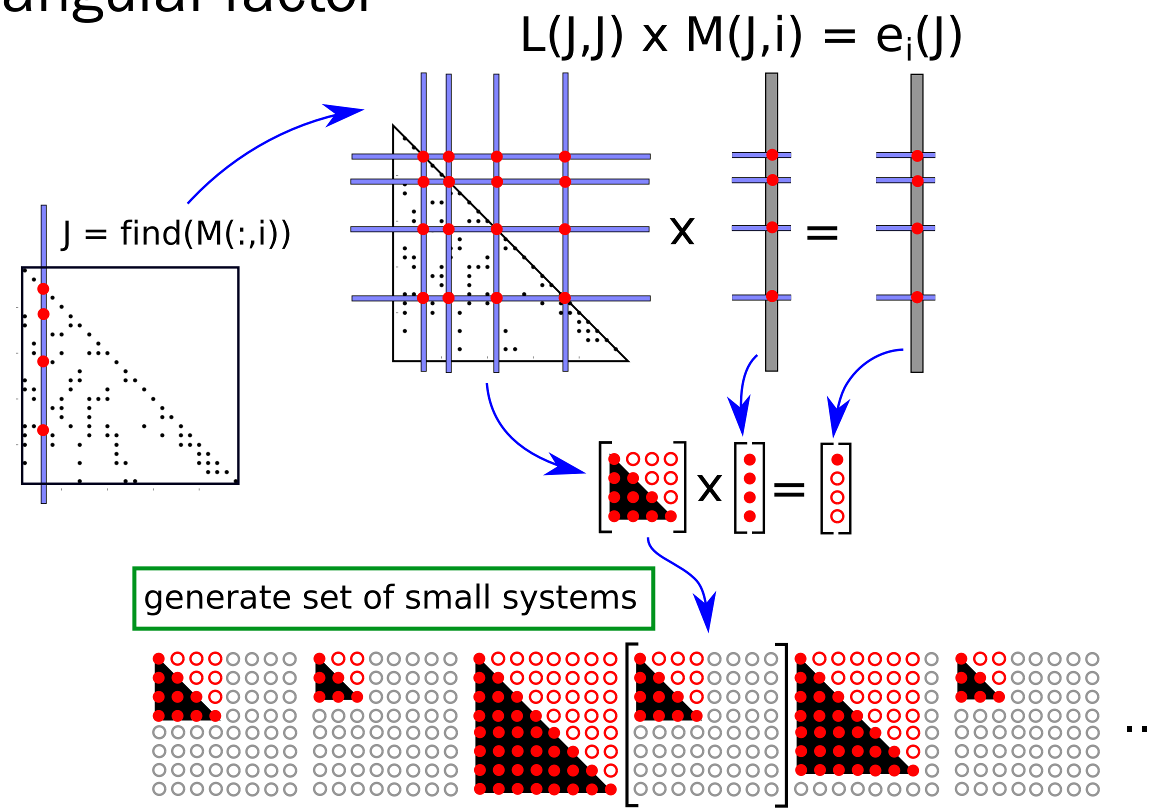

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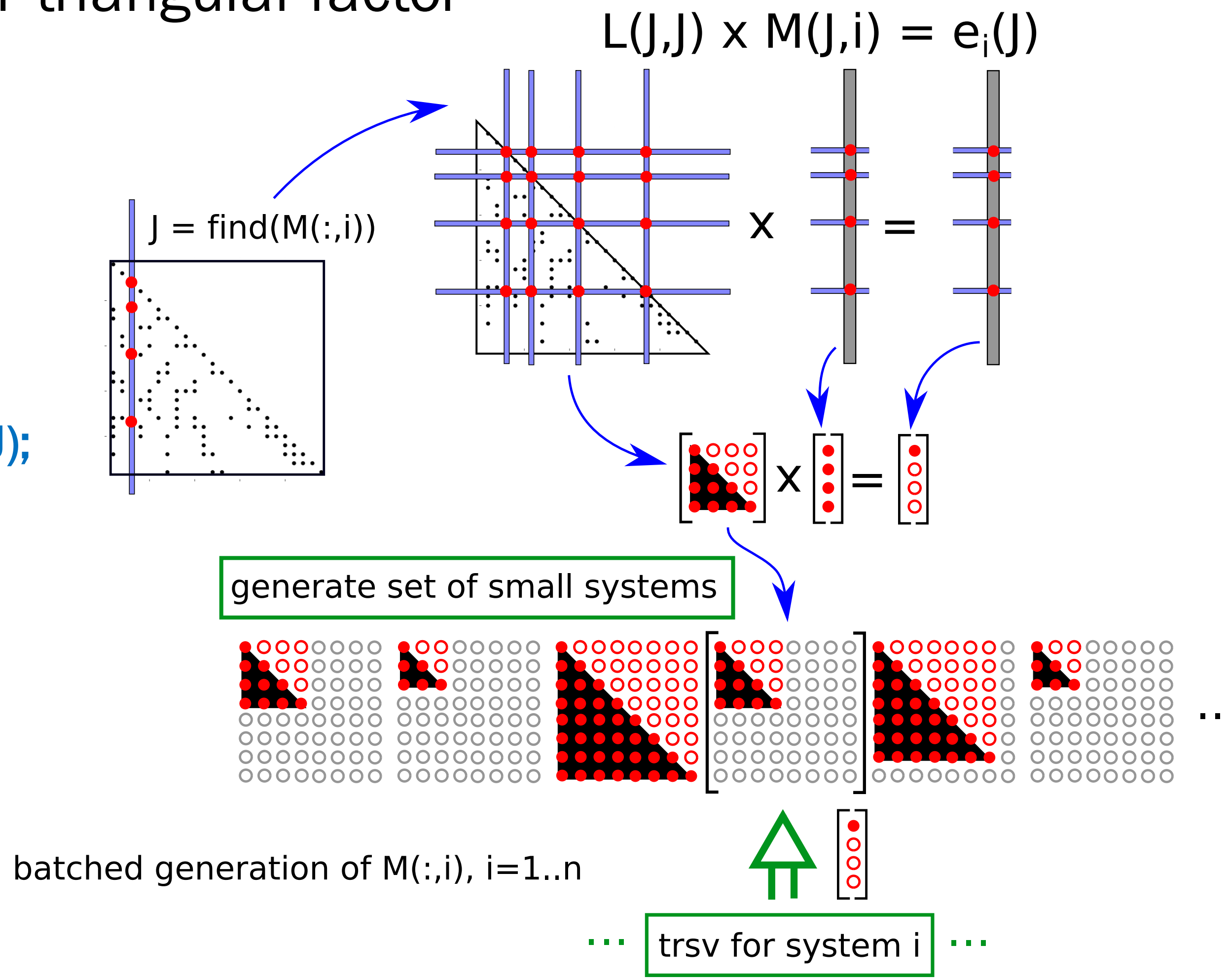

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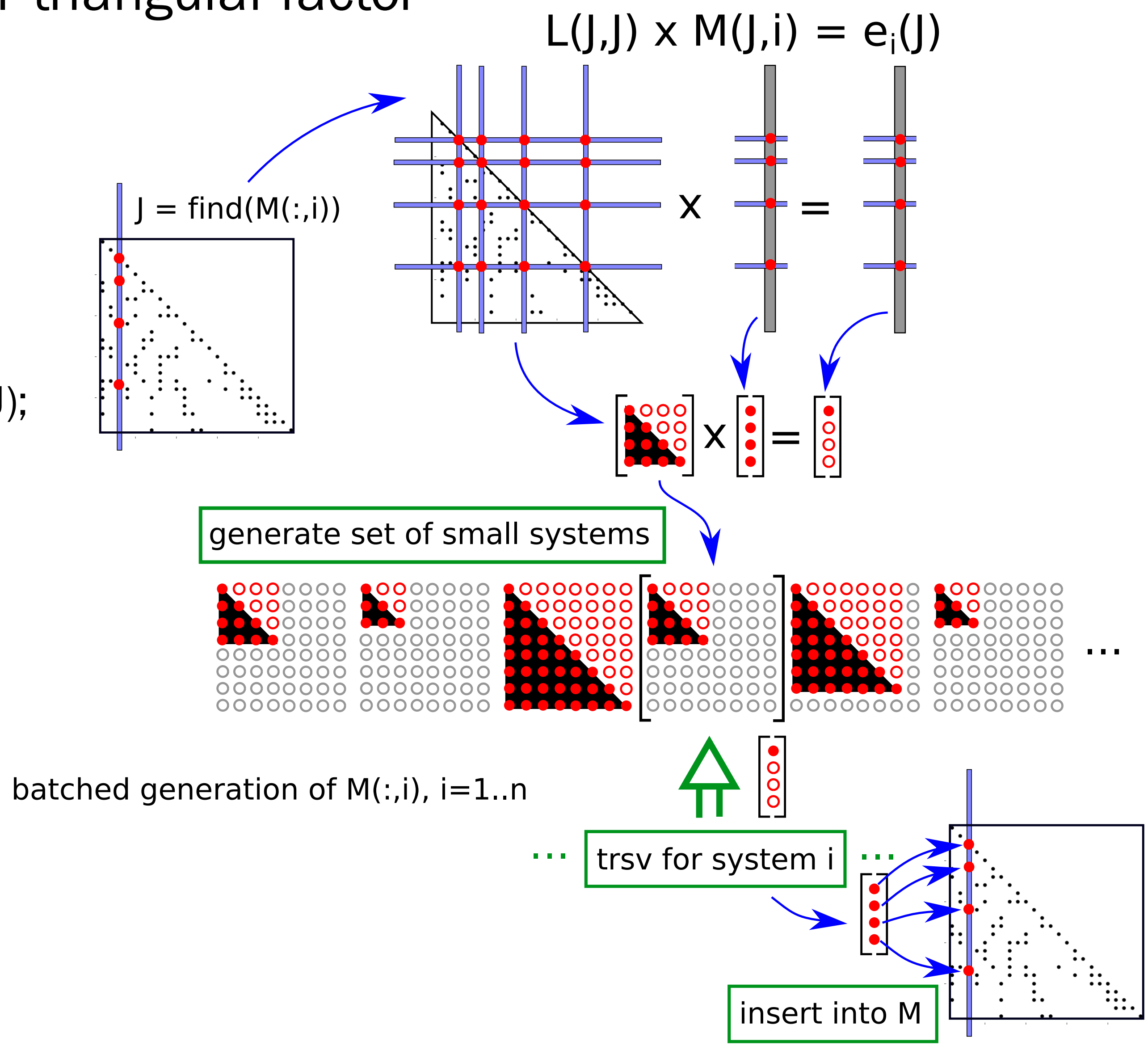

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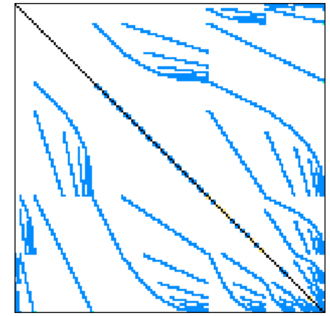
Four Batched Routines:

- Find the locations in each row
  - store size information for small tri-systems
  - store nonzero-locations to find matches
- Generate batch of small triangular systems
  - different sizes in uniformly-sized blocks
- Batched trsv
  - different sizes
  - non-coalescent in memory (uniform blocks)
  - use kernel-switch for hard-coded sizes
- Batched re-insertion into sparse SpAI matrix
  - non-coalescent reads/writes

- Batched trsv will become standard building block.
- Batched routines for extracting/inserting data into sparse structures?

Example: ILU - BiCGSTAB



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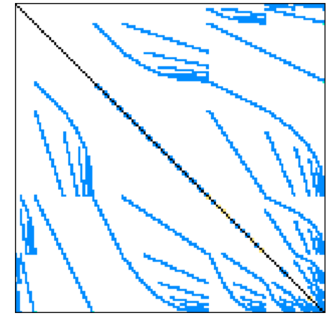

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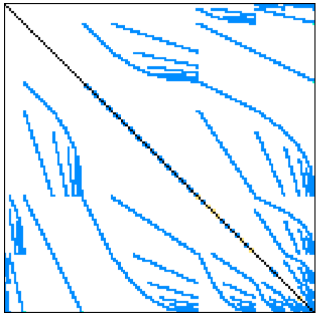

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Example: ILU - BiCGSTAB

Matrix	cuSPARSE		5 Jacobi		SPAI L		SPAI L^2	
Laplace 2D	648	58.04 s	1,263	24.89 s	1,011	4.93 s	743	6.67 s
Laplace 3D	54	4.12 s	85	5.96 s	83	3.64 s	*	*
chipcool0	63	1.94 s	61	0.16 s	102	0.18 s	*	*
apache2	631	22.52 s	-	-	-	-	-	-
ani7	1,356	76.46 s	-	-	5,873	8.94 s	3,178	5.98 s
shallow_water2	9	0.38 s	109	0.41 s	11	0.15 s	9	0.18 s
chem_master1	83	1.35 s	-	-	244	0.24 s	163	0.18 s
stomach	13	1.36 s	14	0.44 s	23	0.50 s	*	*
airfoil2d	50	2.92 s	52	0.16 s	108	0.17 s	*	*
G3_circuit	862	104.22 s	-	-	-	-	-	-
tmt_unsym	?	>10k s	2,455	71.74 s	3,070	40.04 s	2,665	54.34 s
FEM3Dthermal2	6	1.33 s	-	-	21	0.59 s	-	-
venkat01	13	0.99 s	17	0.23 s	38	0.31 s	*	*
mesh96	54	4.63 s	84	5.42 s	83	3.71 s	*	*

* High memory requirement
\- No convergence

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Proof-of-concept study:

- Large set of general matrices from University of Florida Matrix Collection
  UFMC; https://www.cise.ufl.edu/research/sparse/matrices/
- BiCGSTAB as outer solver
- Comparison:
  - exact trsv (cuSPARSE v. 7.5),
  - Jacobi sweeps,
  - SpAI using sparsity pattern of spy(L), spy(L^2), spy(L^3)
- All computational routines from MAGMA-sparse (Nvidia K40 GPU, CUDA 7.5)

http://www.icl.utk.edu/~hanzt/precond_comparison/

Summary and Outlook

SpAI for solving **sparse triangular systems in preconditioning** very efficient:

- SpMV *much faster* than triangular exact solves
- Fast SpAI *generation via batched routines*
- *Flexibility* in choosing the *SpAI nonzero-structure*

Future work

- *Optimize* nonzero-structure (preconditioner quality vs. performance)
- Interplay with *CA-Krylov methods* (precond. communication pattern...)
- SpAI for efficient ILU preconditioning on *distributed systems*

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