A Proposal for Fortran 95 Thin BLAS

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1 Introduction

This paper presents a proposal for a specification of Fortran 95 thin BLAS.

A preliminary version of the F90 Blas proposal has been circulated informally (but not very widely) for about 4 years, and code which implements that version has been available in the Fortran 90 software repository maintained by NAG Ltd (**http://www.nag.co.uk**).

This proposal is designed to cover — as far as seems sensible — roughly the same functionality as the original Level 1, 2 and 3 (Fortran 77) BLAS. It does not address sparse matrices or vectors, nor does it explicitly address issues of parallel computation.

Many of the Fortran 77 Level 1 BLAS can be replaced by simple array expressions and assignments in Fortran 95, without loss of convenience or performance (assuming a reasonable degree of optimisation by the compiler). Fortran 95 also allows groups of related Level 2 and Level 3 BLAS to be merged together, each group being covered by a single interface.

2 Design of Fortran 95 Interfaces

Our proposed design utilizes the following features of the Fortran 95 language.

Generic interfaces: all routines are accessed through *generic* interfaces. A single generic name covers several specific instances whose arguments may differ in the following properties:

data type (real or complex). However, the relevant arguments must be either all real or all complex. (We do not, for example, cater for multiplying a real matrix by a complex matrix, though this functionality could easily be added to the design if there was a need for it.)

precision (or equivalently, kind-value). However, all real or complex arguments must have the same precision.

rank Some arguments may either have rank 2 (to store a matrix) or rank 1 (to store a vector).

different argument list Some of the arguments may not appear in a specific instance. In this case a pre-defined value or a pre-defined action is assumed. The following table contains the pre-defined value or action for the argument that may

not be used. Some of these arguments are key arguments that will be described later.

argument	value or action if the argument is not used
alpha	1.0 or (1.0,0.0)
beta	0.0 or (0.0,0.0)
op_x	operate with x
lower	reference upper triangle only
right_side	operate on the left-hand side
unit_diag	non-unit triangular

Assumed-shape arrays: all array arguments are *assumed-shape* arrays, which must have the exact shape required to store the corresponding matrix or vector. Hence arguments to specify array-dimensions or problem-dimensions are not required. The routines assume that the supplied arrays have valid and consistent shapes (see Section 5). Zero dimensions (implying empty arrays) are allowed.

Key arguments: in the Fortran 77 BLAS, we use character arguments to specify different options for the operation to be performed. In this proposal we suggest using key arguments. A key argument is a dummy argument whose actual argument must be a named constant defined by BLAS. The following table lists the key arguments, the related BLAS named constants and the equivalent F77 BLAS values.

dummy argument	named constant	meaning	F77 argument
op_x	*not used*	operate with x	TRANSx = 'N'
	blas_trans	operate with transpose x	TRANSx = 'T'
	blas_conj	operate with conjugate x	TRANSx = 'C'
	blas_conj_trans	operate with conjugate-transpose x	TRANSx = 'H'
lower	*not used*	reference upper triangle only	UPLO = 'U'
	blas_lower	reference lower triangle only	UPLO = 'L'
right_side	*not used*	operate on the left-hand side	SIDE = 'L'
	blas_right	operate on the right-hand side	SIDE = 'R'
unit_diag	*not used*	non-unit triangular	DIAG = 'N'
	blas_unit_diag	unit triangular	DIAG = 'U'

3 Interfaces for Real Data

The primary aim of this paper is to convey the flavour of the different generic interfaces.

Therefore we first describe the interfaces as they apply to *real* data. The extra complications which arise when they apply to complex data will be considered in Section 4.

We summarize each interface in the form of a `subroutine` statement (or in one case a `function` statement), in which all the arguments might appear. (This is a convenient way to think of the interface, although such a statement using the generic interface name never

appears in the code.) Arguments which need not be supplied are enclosed in square brackets, for example:

```
subroutine trmm( [alpha,] a, [op_a,] b, [lower,] [right_side,] [unit_diag] )
```

This is followed by a table which lists the different variants of the operation, depending either on the ranks of some of the arguments or on the key arguments.

The following table shows the values used in the tables and the related named constant for the key arguments.

dummy argument	value in table	named constant
op_x	'T'	blas_trans
	'C'	blas_conj
	'C/T'	blas_conj_trans
right_side	'R'	blas_right

3.1 Routines using conventional storage for matrices

By conventional storage, we mean storing a matrix in a 2-dimensional array,

```
subroutine gemm( [alpha,] a, [op_a,] b, [op_b], [beta,] c )
```

rank of a	rank of b	rank of c	op_a	op_b	operation	F77 BLAS
2	2	2			$C \leftarrow \alpha AB + \beta C$	_GEMM
2	2	2		'T'	$C \leftarrow \alpha AB^T + \beta C$	_GEMM
2	2	2	'T'		$C \leftarrow \alpha A^T B + \beta C$	_GEMM
2	2	2	'T'	'T'	$C \leftarrow \alpha A^T B^T + \beta C$	_GEMM
2	1	1			$c \leftarrow \alpha Ab + \beta c$	_GEMV
2	1	1	'T'		$c \leftarrow \alpha A^T b + \beta c$	_GEMV
1	1	2			$C \leftarrow \alpha ab^T + \beta C$	_GER_

```
subroutine symm( [alpha,] a, b, [beta,] c, [lower,] [right_side] )
subroutine hemm( [alpha,] a, b, [beta,] c, [lower,] [right_side] )
```

rank of b	rank of c	right_side	operation	F77 BLAS
2	2		$C \leftarrow \alpha AB + \beta C$	_SYMM
2	2	'R'	$C \leftarrow \alpha BA + \beta C$	_SYMM
1	1		$c \leftarrow \alpha Ab + \beta c$	_SYMV

where A is a symmetric matrix.

```
subroutine syrk( [alpha,] a, [op_a,] [beta,] c, [lower] )
subroutine herk( [alpha,] a, [op_a,] [beta,] c, [lower] )
```

rank of a	op_a	operation	F77 BLAS
2		$C \leftarrow \alpha A A^T + \beta C$	_SYRK
2	'T'	$C \leftarrow \alpha A^T A + \beta C$	_SYRK
1		$C \leftarrow \alpha a a^T + \beta C$	_SYR1

where C is a symmetric matrix.

```
subroutine syr2k( [alpha,] a, [op_a,] b, [beta,] c, [lower] )
subroutine he2rk( [alpha,] a, [op_a,] b, [beta,] c, [lower] )
```

rank of a	rank of b	op_a	operation	F77 BLAS
2	2		$C \leftarrow \alpha A B^T + \alpha B A^T + \beta C$	_SYR2K
2	2	'T'	$C \leftarrow \alpha A^T B + \alpha B^T A + \beta C$	_SYR2K
1	1		$C \leftarrow \alpha a b^T + \alpha b a^T + \beta C$	_SYR2

where C is a symmetric matrix.

```
subroutine trmm( [alpha,] a, [op_a,] b, [lower,] [right_side,] [unit_diag] )
```

rank of b	op_a	right_side	operation	F77 BLAS
2			$B \leftarrow \alpha A B$	_TRMM
2	'T'		$B \leftarrow \alpha A^T B$	_TRMM
2		'R'	$B \leftarrow \alpha B A$	_TRMM
2	'T'	'R'	$B \leftarrow \alpha B A^T$	_TRMM
1			$b \leftarrow \alpha A b$	_TRMV
1	'T'		$b \leftarrow \alpha A^T b$	_TRMV

```
subroutine trsm( [alpha,] a, [op_a,] b, [lower,] [right_side,] [unit_diag] )
```

rank of b	op_a	right_side	operation	F77 BLAS
2			$B \leftarrow \alpha A^{-1} B$	_TRSM
2	'T'		$B \leftarrow \alpha A^{-T} B$	_TRSM
2		'R'	$B \leftarrow \alpha B A^{-1}$	_TRSM
2	'T'	'R'	$B \leftarrow \alpha B A^{-T}$	_TRSM
1			$b \leftarrow \alpha A^{-1} b$	_TRSV
1	'T'		$b \leftarrow \alpha A^{-T} b$	_TRSV

3.2 Routines using packed storage for matrices

By *packed* storage, we mean storing the upper or lower triangle of a symmetric or triangular matrix in a 1-dimensional array (i.e. a vector).

```
subroutine spmv( [alpha,] a, b, [beta,] c, [lower] )
subroutine hpmv( [alpha,] a, b, [beta,] c, [lower] )
```

operation	F77 BLAS
$c \leftarrow \alpha Ab + \beta c$	_SPMV

where A is a symmetric matrix.

```
subroutine spr1( [alpha,] a, [beta,] c, [lower] )
subroutine hpr1( [alpha,] a, [beta,] c, [lower] )
```

operation	F77 BLAS
$C \leftarrow \alpha aa^T + \beta C$	_SPR1

where C is a symmetric matrix.

```
subroutine syr2( [alpha,] a, b, [beta,] c, [lower] )
subroutine he2r( [alpha,] a, b, [beta,] c, [lower] )
```

operation	F77 BLAS
$C \leftarrow \alpha ab^T + \alpha ba^T + \beta C$	_SYR2

where C is a symmetric matrix.

```
subroutine tpmv( [alpha,] a, [op_a,] b, [lower,] [unit_diag] )
```

op_a	operation	F77 BLAS
	$b \leftarrow \alpha Ab$	_TPMV
'T'	$b \leftarrow \alpha A^T b$	_TPMV

where A is a triangular matrix.

```
subroutine tpsv( [alpha,] a, [op_a,] b, [lower,] [unit_diag] )
```

op_a	operation	F77 BLAS
	$b \leftarrow \alpha A^{-1} b$	_TPSV
'T'	$b \leftarrow \alpha A^{-T} b$	_TPSV

where A is a triangular matrix.

3.3 Routines for band matrices

`subroutine gbmv( [alpha,] a, [op_a,] b, [beta,] c, kd )`

op_a	operation	F77 BLAS
	$C \leftarrow \alpha Ab + \beta c$	_GBMV
'T'	$C \leftarrow \alpha A^T b + \beta c$	_GBMV

where A is a general band matrix with `kd` superdiagonals supplied.

`subroutine sbmv( [alpha,] a, b, [beta,] c, [lower] )`
`subroutine hbmv( [alpha,] a, b, [beta,] c, [lower] )`

operation	F77 BLAS
$c \leftarrow \alpha Ab + \beta c$	_SPMV

where A is a symmetric band matrix.

`subroutine tbmv( [alpha,] a, [op_a,] , [lower,] [unit_diag] )`

op_a	operation	F77 BLAS
	$b \leftarrow \alpha Ab$	_TBMV
'T'	$b \leftarrow \alpha A^T b$	_TBMV

where A is a triangular band matrix.

`subroutine tpsv( [alpha,] a, [op_a,] , [lower,] [unit_diag] )`

op_a	operation	F77 BLAS
	$b \leftarrow \alpha A^{-1} b$	_TBSV
'T'	$b \leftarrow \alpha A^{-T} b$	_TBSV

where A is a triangular band matrix.

3.4 Level 1 routines

`function nrm2( x )`

Operation: return $\|x\|_2$.

`subroutine swap( x, y )`

Operation: $x \leftrightarrow y$.

`subroutine rot( x, y, c, s )`

`subroutine rotg( a, b, c, s )`

4 Interfaces for Complex Data

In this section we show the subroutine `gemm` for complex arguments. The generic interface is that described for real arguments.

rank of `a`	rank of `b`	rank of `c`	`op_a`	`op_b`	operation	F77 BLAS
2	2	2			$C \leftarrow \alpha AB + \beta C$	`_GEMM`
2	2	2		'T'	$C \leftarrow \alpha AB^T + \beta C$	`_GEMM`
2	2	2		'C/T'	$C \leftarrow \alpha AB^H + \beta C$	`_GEMM`
2	2	2	'T'		$C \leftarrow \alpha A^T B + \beta C$	`_GEMM`
2	2	2	'T'	'T'	$C \leftarrow \alpha A^T B^T + \beta C$	`_GEMM`
2	2	2	'T'	'C/T'	$C \leftarrow \alpha A^T B^H + \beta C$	`_GEMM`
2	2	2	'C/T'		$C \leftarrow \alpha A^H B + \beta C$	`_GEMM`
2	2	2	'C/T'	'T'	$C \leftarrow \alpha A^H B^T + \beta C$	`_GEMM`
2	2	2	'C/T'	'C/T'	$C \leftarrow \alpha A^H B^H + \beta C$	`_GEMM`
2	1	1			$c \leftarrow \alpha Ab + \beta c$	`_GEMV`
2	1	1	'T'		$c \leftarrow \alpha A^T b + \beta c$	`_GEMV`
2	1	1	'C/T'		$c \leftarrow \alpha A^H b + \beta c$	`_GEMV`
1	1	2			$C \leftarrow \alpha ab^T + \beta C$	`_GERU`
1	1	2		'C'	$C \leftarrow \alpha ab^H + \beta C$	`_GERC`

5 Error checking

We propose that the Fortran 95 thin BLAS perform no checks on their arguments.

6 Comparison with the Fortran 77 BLAS

We consider in more detail each Level of BLAS in turn. In referring to particular operations performed by the BLAS, we use the traditional BLAS names, except that we omit the initial letter (S, D, C, Z) which indicates the data type — for example, `SWAP`. The resulting names are also the generic names which we propose for the Fortran 95 interfaces.

6.1 Level 1

We include in this proposal only the following:

```
SWAP
ROT
NRM2
ROTG
```

`ROT` and `ROTG` have been extended to cover complex rotations.

6.2 Level 2

We propose to combine many of the Level 2 BLAS with the corresponding Level 3 BLAS in a single generic interface, the different instances being distinguished by the ranks of some of the arguments. In order to do this, we propose to remove some minor inconsistencies between the specifications of the Level 2 and Level 3 routines:

We propose adding one new routine:

```
REFG
```

to generate an elementary reflector (that is, a Householder matrix), following the same specification as the LAPACK auxiliary routine xLARTG.

The scope of the proposed BLAS has been extended slightly compared with the Fortran 77 BLAS: for example, we propose Level 1 BLAS for generating an elementary reflector (Householder matrix), and for generating and applying complex plane rotations; we also propose Level 2 BLAS for complex symmetric matrices. On the other hand, many of the Fortran 77 Level 1 BLAS can be replaced in Fortran 95 by simple array constructs, and they have been omitted.

For the thin BLAS we propose that the code does not do any checks on the arguments.

We propose generic interfaces that cover — wherever relevant — both Level 2 and Level 3 BLAS (for example, xTRSV and xTRSM), and have modified the specification of some Level 2 BLAS to make them more consistent with the Level 3 BLAS (for example, xTRSV now has an argument `alpha`).

For each procedure we specify a number of arguments that must be supplied and another set of arguments that need not be supplied. We specify a value or action for each argument which need not be supplied.

We propose that the thin BLAS contain a specific instance for each possible case and no checks or branching is used within the code.

We propose that the early implementations for the thin BLAS will contain simple calls to the reliable and tested F77 BLAS.

For example, the generic `gemm` will consist of the following specific procedures:

- 36 specific procedures each of which calls the F77 BLAS procedure `ZGEMM` (3 settings for each of `op_a` and `op_b`, and 2 settings for each of `alpha` and `beta`).
- 12 specific procedures each of which calls the F77 BLAS procedure `ZGEMV` (3 settings for `op_a`, and 2 settings for each of `alpha` and `beta`).
- 4 specific procedures each of which calls the F77 BLAS procedure `ZGERU` (2 settings for each of `alpha` and `beta`).

- 4 specific procedures each of which calls the F77 BLAS procedure `ZGERC` (2 settings for each of `alpha` and `beta`).

- 36 specific procedures each of which calls the F77 BLAS procedure `DGEMM` (3 settings for each of `op_a` and `op_b`, and 2 settings for each of `alpha` and `beta`). Only 16 procedures are needed, but we allow for `op_a = blas_conj_trans` for similarity with the complex case.

- 12 specific procedures each of which calls the F77 BLAS procedure `DGEMV` (3 settings for `op_a`, and 2 settings for each of `alpha` and `beta`). Only 8 procedures are needed, but we allow for `op_a = blas_conj_trans` for similarity with the complex case.

- 4 specific procedures each of which calls the F77 BLAS procedure `DGER` (2 settings for each of `alpha` and `beta`).

- 4 specific procedures each of which calls the F77 BLAS procedure `DGER` (2 settings for each of `alpha` and `beta`). These are similar to the previous case but have been added to allow `op_b = blas_conj` (as in the complex case for `ZGERC`).

Appendix A contains a list of these specific procedures (only double precision procedures are listed).

A proposed document for this procedure is given in a separate document.

7 Conclusion

Our principal purpose in presenting this specification at this meeting is to provide additional input to the discussion about different levels of genericity in the interface to linear algebra routines. The thin BLAS are designed principally as building-blocks for software developers and for the BLAS itself.

A Appendix: Specific procedures for gemm

This appendix contains a list of the specific procedures for the generic procedure gemm.

```
!
! 36 procedures each calls the F77 BLAS subroutine ZGEMM
!  a, b and c are rank-2
!
!            alpha      op_a       a     op_b        b beta c   operation
!
! zgemm_301 (alpha,blas_conj_trans,a,blas_conj_trans,b,beta,c)  C < alpha A(H) B(H) + beta C
! zgemm_302 (alpha,blas_conj_trans,a,blas_conj_trans,b,     c)  C < alpha A(H) B(H) + C
! zgemm_303 (alpha,blas_conj_trans,a,blas_trans     ,b,beta,c)  C < alpha A(H) B(T) + beta C
! zgemm_304 (alpha,blas_conj_trans,a,blas_trans     ,b,     c)  C < alpha A(H) B(T) + C
! zgemm_305 (alpha,blas_conj_trans,a,               b,beta,c)  C < alpha A(H) B + beta C
! zgemm_306 (alpha,blas_conj_trans,a,               b,     c)  C < alpha A(H) B + C
! zgemm_307 (alpha,blas_trans     ,a,blas_conj_trans,b,beta,c)  C < alpha A(T) B(H) + beta C
! zgemm_308 (alpha,blas_trans     ,a,blas_conj_trans,b,     c)  C < alpha A(T) B(H) + C
! zgemm_309 (alpha,blas_trans     ,a,blas_trans     ,b,beta,c)  C < alpha A(T) B(T) + beta C
! zgemm_310 (alpha,blas_trans     ,a,blas_trans     ,b,     c)  C < alpha A(T) B(T) + C
! zgemm_311 (alpha,blas_trans     ,a,               b,beta,c)  C < alpha A(T) B + beta C
! zgemm_312 (alpha,blas_trans     ,a,               b,     c)  C < alpha A(T) B + C
! zgemm_313 (alpha,               a,blas_conj_trans,b,beta,c)  C < alpha A B(H) + beta C
! zgemm_314 (alpha,               a,blas_conj_trans,b,     c)  C < alpha A B(H) + C
! zgemm_315 (alpha,               a,blas_trans     ,b,beta,c)  C < alpha A B(T) + beta C
! zgemm_316 (alpha,               a,blas_trans     ,b,     c)  C < alpha A B(T) + C
! zgemm_317 (alpha,               a,               b,beta,c)  C < alpha A B + beta C
! zgemm_318 (alpha,               a,               b,     c)  C < alpha A B + C
! zgemm_319 (      blas_conj_trans,a,blas_conj_trans,b,beta,c)  C < A(H) B(H) + beta C
! zgemm_320 (      blas_conj_trans,a,blas_conj_trans,b,     c)  C < A(H) B(H) + C
! zgemm_321 (      blas_conj_trans,a,blas_trans     ,b,beta,c)  C < A(H) B(T) + beta C
! zgemm_322 (      blas_conj_trans,a,blas_trans     ,b,     c)  C < A(H) B(T) + C
! zgemm_323 (      blas_conj_trans,a,               b,beta,c)  C < A(H) B + beta C
! zgemm_324 (      blas_conj_trans,a,               b,     c)  C < A(H) B + C
! zgemm_325 (      blas_trans     ,a,blas_conj_trans,b,beta,c)  C < A(T) B(H) + beta C
! zgemm_326 (      blas_trans     ,a,blas_conj_trans,b,     c)  C < A(T) B(H) + C
! zgemm_327 (      blas_trans     ,a,blas_trans     ,b,beta,c)  C < A(T) B(T) + beta C
! zgemm_328 (      blas_trans     ,a,blas_trans     ,b,     c)  C < A(T) B(T) + C
! zgemm_329 (      blas_trans     ,a,               b,beta,c)  C < A(T) B + beta C
! zgemm_330 (      blas_trans     ,a,               b,     c)  C < A(T) B + C
! zgemm_331 (                      a,blas_conj_trans,b,beta,c)  C < A B(H) + beta C
! zgemm_332 (                      a,blas_conj_trans,b,     c)  C < A B(H) + C
! zgemm_333 (                      a,blas_trans     ,b,beta,c)  C < A B(T) + beta C
! zgemm_334 (                      a,blas_trans     ,b,     c)  C < A B(T) + C
! zgemm_335 (                      a,               b,beta,c)  C < A B + beta C
! zgemm_336 (                      a,               b,     c)  C < A B + C
!
!
! 12 procedures each calls the F77 BLAS subroutine ZGEMV
!  a is rank-2, and b and c are rank-1
!
!            alpha      op_a       a     op_b        b beta c   operation
!
! zgemv_201 (alpha,blas_conj_trans,a,               b,beta,c)  c < alpha A(H) b + beta c
! zgemv_202 (alpha,blas_conj_trans,a,               b,     c)  c < alpha A(H) b + c
! zgemv_203 (alpha,blas_trans     ,a,               b,beta,c)  c < alpha A(T) b + beta c
! zgemv_204 (alpha,blas_trans     ,a,               b,     c)  c < alpha A(T) b + c
! zgemv_205 (alpha,               a,               b,beta,c)  c < alpha A b + beta c
! zgemv_206 (alpha,               a,               b,     c)  c < alpha A b + c
! zgemv_207 (      blas_conj_trans,a,               b,beta,c)  c < A(H) b + beta c
! zgemv_208 (      blas_conj_trans,a,               b,     c)  c < A(H) b + c
! zgemv_209 (      blas_trans     ,a,               b,beta,c)  c < A(T) b + beta c
! zgemv_200 (      blas_trans     ,a,               b,     c)  c < A(T) b + c
! zgemv_211 (                      a,               b,beta,c)  c < A b + beta c
! zgemv_212 (                      a,               b,     c)  c < A b + c
!
```

```
!
! 4 procedures each calls the F77 BLAS subroutine ZGERU
!  c is rank-2, and a and b are rank-1
!
!            alpha      op_a       a     op_b        b beta c   operation
!
! zgeru_201 (alpha,               a,               b,beta,c)  C < alpha a b(T) + beta C
! zgeru_202 (alpha,               a,               b,     c)  C < alpha a b(T) + C
! zgeru_203 (                     a,               b,beta,c)  C < a b(T) + beta C
! zgeru_204 (                     a,               b,     c)  C < a b(T) + C
!
!
! 4 procedures each calls the F77 BLAS subroutine ZGERC
!  c is rank-2, and a and b are rank-1
!
!            alpha      op_a       a     op_b        b beta c   operation
!
! zgerc_201 (alpha,blas_conj      a,               b,beta,c)  C < alpha a b(H) + beta C
! zgerc_202 (alpha,blas_conj      a,               b,     c)  C < alpha a b(H) + C
! zgerc_203 (      blas_conj      a,               b,beta,c)  C < a b(H) + beta C
! zgerc_204 (      blas_conj      a,               b,     c)  C < a b(H) + C
!
!
!
! 36 procedures each calls the F77 BLAS subroutine DGEMM
!  a, b and c are rank-2
!
!            alpha      op_a       a     op_b        b beta c   operation
!
! dgemm_301 (alpha,blas_conj_trans,a,blas_conj_trans,b,beta,c)  C < alpha A(H) B(H) + beta C
! dgemm_302 (alpha,blas_conj_trans,a,blas_conj_trans,b,     c)  C < alpha A(H) B(H) + C
! dgemm_303 (alpha,blas_conj_trans,a,blas_trans     ,b,beta,c)  C < alpha A(H) B(T) + beta C
! ...
! ...
! dgemm_334 (                     a,blas_trans     ,b,     c)  C < A B(T) + C
! dgemm_335 (                     a,               b,beta,c)  C < A B + beta C
! dgemm_336 (                     a,               b,     c)  C < A B + C
!
!
! 12 procedures each calls the F77 BLAS subroutine DGEMV
!  a is rank-2, and b and c are rank-1
!
!            alpha      op_a       a     op_b        b beta c   operation
!
! dgemv_201 (alpha,blas_conj_trans,a,               b,beta,c)  c < alpha A(H) b + beta c
! dgemv_202 (alpha,blas_conj_trans,a,               b,     c)  c < alpha A(H) b + c
! ...
! dgemv_211 (                     a,               b,beta,c)  c < A b + beta c
! dgemv_212 (                     a,               b,     c)  c < A b + c
!
!
! 8 procedures each calls the F77 BLAS subroutine DGER
!  c is rank-2, and a and b are rank-1
!
!            alpha      op_a       a     op_b        b beta c   operation
!
! dger_201  (alpha,               a,               b,beta,c)  C < alpha a b(T) + beta C
! dger_202  (alpha,               a,               b,     c)  C < alpha a b(T) + C
! dger_203  (                     a,               b,beta,c)  C < a b(T) + beta C
! dger_204  (                     a,               b,     c)  C < a b(T) + C
! dger_205  (alpha,blas_conj      a,               b,beta,c)  C < alpha a b(H) + beta C
! dger_206  (alpha,blas_conj      a,               b,     c)  C < alpha a b(H) + C
! dger_207  (      blas_conj      a,               b,beta,c)  C < a b(H) + beta C
! dger_208  (      blas_conj      a,               b,     c)  C < a b(H) + C
```