11.3 Conversion between Chebyshev and Monomial Representations of a Polynomial

A. Purpose

These subroutines convert a polynomial represented in the monomial basis to a representation in the Chebyshev basis, and vice versa.

B. Usage

B.1 Program Prototype, Single Precision

INTEGER N
REAL COEFF(0:≥N)

Assign values to N, and to coefficients in COEFF(). If COEFF(i) contains coefficients of $T_i(x)$, $i = 0, 1, \ldots, N$, which are to be converted to coefficients of $x^i$.

CALL SCONCM(N, COEFF)

For the inverse operation,

CALL SCONMC(N, COEFF)

B.1 Argument Definitions

N [in] The degree of the polynomial.

COEFF [inout] When calling SCONCM, COEFF(i) contains the coefficient of $T_i(x)$, $i = 0, 1, \ldots, N$, on input, and contains the coefficient of $x^i$ on output. When calling SCONMC, COEFF(i) contains the coefficient of $x^i$, $i = 0, 1, \ldots, N$, on input, and the coefficient of $T_i$ on output.

B.2 Modifications for Double Precision

Change the names SCONCM and SCONMC to DCONCM and DCONMC respectively, and change the REAL declaration to DOUBLE PRECISION.

C. Examples and Remarks

The program DRSCON prints out the coefficients of the Chebyshev polynomials corresponding to $x^k$, $k = 0, 1, \ldots, 6$, and then prints the coefficients in the monomial basis corresponding to the Chebyshev polynomials $T_k$, $k = 0, 1, \ldots, 6$. Results are in the file ODSCON.

If these subroutines are applied to a coefficient array, say P(), obtained from SPFIT, Chapter 11.1, the zero-th order coefficient is in P(3) so the call would be of the form SCONxx(NDEG, P(3)), where xx is either CM or MC.

D. Functional Description

Consider the polynomial $p_n(x)$ of degree $n$,

$$ p_n(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} c_k T_k(x) \quad (1) $$

where $T_k(x)$ is the $k$th Chebyshev polynomial. This software converts between the $a_k$'s and the $c_k$'s.

Using the well-known identities,

$$ xT_k(x) = \frac{1}{2} [T_{k+1}(x) + T_{k-1}(x)], \quad k > 1 $$
$$ xT_0(x) = T_1(x) = x, $$

we can write $p_n$ in forms intermediate between the extremes represented in Eq. (1). It is these intermediate forms that are used in obtaining the recurrences. Thus

$$ p_n(x) = \sum_{k=0}^{j-1} a_k x^k + x^j \sum_{k=0}^{n-j} b_{k,j} T_k(x) \quad (3) $$

$$ \equiv \sum_{k=0}^{j} a_k x^k + x^{j+1} \sum_{k=0}^{n-j-1} b_{k,j+1} T_k(x) \quad (4) $$

Note that $b_{k,0} \equiv c_k$. Using Eq. (2), Eq. (4) gives

$$ p_n(x) = \sum_{k=0}^{j} a_k x^k + x^{j+1} \sum_{k=0}^{n-j-1} b_{k,j+1} T_k(x) + T_{k-1}(x)] + x^2 b_{0,j+1} T_1(x). \quad (5) $$

Collecting like terms in Eqs. (3) and (5), we obtain,

$$ a_j + \frac{1}{2} b_{1,j+1} = b_{0,j} $$
$$ b_{0,j+1} + \frac{1}{2} b_{2,j+1} = b_{1,j} $$

$$ \frac{1}{2} [b_{k-1,j+1} + b_{k+1,j+1}] = b_{k,j}, \quad k = 2, 3, \ldots, n - j - 2 $$
$$ \frac{1}{2} b_{k-1,j+1} = b_{k,j}, \quad k \geq n - j - 1. \quad (6) $$

A more efficient recursion is obtained with $b_{k,j}$ replaced by $2^j B_{k,j}$. Thus,

$$ 2^{-j} a_j + B_{1,j+1} = B_{0,j} $$
$$ 2 B_{0,j+1} + B_{2,j+1} = B_{1,j} $$

$$ B_{k-1,j+1} + B_{k+1,j+1} = B_{k,j}, \quad k = 2, 3, \ldots, n - j - 2 $$
$$ B_{k-1,j+1} = B_{k,j}, \quad k \geq n - j - 1. \quad (7) $$

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In the code, the $B_{k,j,k}$ share space with the original $a_k$ or the original $c_k$. If one starts with the $a_k$ then one runs $j$ from $n$ down to 0, and otherwise $j$ runs in the opposite direction. Observe that the innermost loop requires only a single addition.

E. Error Procedures and Restrictions

If $n < 0$, a return is made without taking any action.

F. Supporting Information


<table>
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<tr>
<th>Entry</th>
<th>Required Files</th>
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<tbody>
<tr>
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DRSCON

program DRSCON

$\ldots$

integer NMAX
parameter (NMAX=6)
integer K, N
real COEFF(0:NMAX)

$\ldots$

c% printf( "")
c% for (k = 0; k <= NMAX; k++) printf( "X**%ld", k );
c% printf( "\n") ;
print ("(7X, 9(:'X**' , I1 )', (K, K = 0, NMAX)
do 20 N = 0, NMAX
 do 10 K = 0, N-1
 COEFF(K) = 0.E0
 10 continue
 COEFF(N) = 1.E0
call SCONCM(N, COEFF)
 print ("( ' T' ', I1 , ' (X) = ',' , F7.3, 8F8.3 ')', N,
( COEFF(K), K = 0, N)
do 120 N = 0, NMAX
 do 110 K = 0, N-1
 COEFF(K) = 0.E0
 110 continue
 COEFF(N) = 1.E0
call SCONMC(N, COEFF)
 print ("( 'X**' ', I1 , ' ' = ',' , 9F8.5 ')', N, (COEFF(K), K = 0, N)
do 120 continue
 stop
end

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### ODSCON

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$T_0(X)$, $T_1(X)$, $T_2(X)$, $T_3(X)$, $T_4(X)$, $T_5(X)$, $T_6(X)$

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