# Relative Perturbation Bounds for the Unitary Polar Factor *

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**Abstract**

Let $B$ be an $m \times n$ ($m \geq n$) complex matrix. It is known that there is a unique *polar decomposition* $B = QH$, where $Q^*Q = I$, the $n \times n$ identity matrix, and $H$ is positive definite, provided $B$ has full column rank. Existing perturbation bounds for complex matrices suggest that in the worst case, the change in $Q$ be proportional to the reciprocal of the smallest singular value of $B$. However, there are situations where this unitary polar factor is much more accurately determined by the data than the existing perturbation bounds would indicate. In this paper the following question is addressed: how much may $Q$ change if $B$ is perturbed to $\widetilde{B} = D_1^* B D_2$? Here $D_1$ and $D_2$ are nonsingular and close to the identity matrices of suitable dimensions. It will be proved that for such kinds of perturbations, the change in $Q$ is bounded only by the distances from $D_1$ and $D_2$ to identity matrices, and thus independent of the singular values of $B$.

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Let $B$ be an $m \times n$ $(m \geq n)$ complex matrix. It is known that there are $Q$ with orthonormal column vectors, i.e., $Q^*Q = I$, and a unique positive semidefinite $H$ such that

$$B = QH. \tag{1}$$

Hereafter $I$ denotes an identity matrix with appropriate dimensions which should be clear from the context or specified. The decomposition (1) is called the *polar decomposition* of $B$. If, in addition, $B$ has full column rank, then $Q$ is uniquely determined also. In fact,

$$H = (B^*B)^{1/2}, \quad Q = B(B^*B)^{-1/2}, \tag{2}$$

where superscript "*" denotes conjugate transpose. The decomposition (1) can also be computed from the *singular value decomposition* (SVD) $B = U\Sigma V^*$ by

$$H = V\Sigma_1 V^*, \quad Q = U_1 V^*, \tag{3}$$

where $U = (U_1, U_2)$ and $V$ are unitary, $U_1$ is $m \times n$, $\Sigma = \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix}$ and $\Sigma_1 = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_n)$ is nonnegative.

There are several published bounds stating how much the two factor matrices $Q$ and $H$ may change if entries of $B$ are perturbed in arbitrary manner [2, 3, 6, 8, 9, 11, 12, 13, 14]. In these papers, no assumption was made on how $B$ was perturbed to $\widetilde{B}$ except possibly an assumption on the smallness of $\|\widetilde{B} - B\|$ for some matrix norm $\|\cdot\|$. Roughly speaking, bounds in these published papers suggest that in the worst case, the change in $Q$ be proportional to the reciprocal of the smallest singular value of $B$.

In this paper, on the other hand, we study the change in $Q$, assuming $B$ is complex and perturbed to $\widetilde{B} = D_1^* B D_2$, where $D_1$ and $D_2$ are nonsingular and close to the identity matrices of suitable dimensions. Such perturbations covers, for example, component-wise relative perturbations to entries of symmetric tridiagonal matrices with zero diagonal [5, 7], entries of bidiagonal and biacyclic matrices [1, 4, 5]. Our results indicate that under such perturbations, the change in $Q$ is independent of the smallest singular value of $B$.

Assume that $B$ has full column rank and so does $\widetilde{B} = D_1^* B D_2$. Let

$$B = QH, \quad \widetilde{B} = \widetilde{Q}\widetilde{H} \tag{4}$$

be the polar decompositions of $B$ and $\widetilde{B}$ respectively, and let

$$B = U\Sigma V^*, \quad \widetilde{B} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* \tag{5}$$

be the SVDs of $B$ and $\widetilde{B}$, respectively, where $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2)$, $\widetilde{U}_1$ is $m \times n$, and $\widetilde{\Sigma} = \begin{pmatrix} \widetilde{\Sigma}_1 \\ 0 \end{pmatrix}$ and $\widetilde{\Sigma}_1 = \mathrm{diag}\,(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$. Assume as usual that

$$\sigma_1 \geq \cdots \geq \sigma_n > 0, \quad \text{and} \quad \widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_n > 0. \tag{6}$$

It follows from (2) and (5) that $Q = U_1 V^*$ and $\widetilde{Q} = \widetilde{U}_1 \widetilde{V}^*$.

In what follows, $\|X\|_{\mathrm{F}}$ denotes the Frobenius norm which is the square root of the trace of $X^*X$. Notice that

$$\begin{aligned}
\widetilde{U}^*(\widetilde{B}-B)V &= \widetilde{\Sigma}\widetilde{V}^*V-\widetilde{U}^*U\Sigma, \\
\widetilde{U}^*(\widetilde{B}-B)V &= \widetilde{U}^*(D_1^*BD_2-D_1^*B+D_1^*B-B)V \\
&= \widetilde{U}^*\left[\widetilde{B}(I-D_2^{-1})+(D_1^*-I)B\right]V \\
&= \widetilde{\Sigma}\widetilde{V}^*(I-D_2^{-1})V+\widetilde{U}^*(D_1^*-I)U\Sigma,
\end{aligned}$$

and

$$\begin{aligned}
U^*(\widetilde{B}-B)\widetilde{V} &= U^*\widetilde{U}\widetilde{\Sigma}-\Sigma V^*\widetilde{V}, \\
U^*(\widetilde{B}-B)\widetilde{V} &= U^*(D_1^*BD_2-BD_2+BD_2-B)\widetilde{V} \\
&= U^*\left[(I-D_1^{-*})\widetilde{B}+B(D_2-I)\right]\widetilde{V} \\
&= U^*(I-D_1^{-*})\widetilde{U}\widetilde{\Sigma}+\Sigma V^*(D_2-I)\widetilde{V}.
\end{aligned}$$

to obtain two perturbation equations:

$$\widetilde{\Sigma}\widetilde{V}^*V-\widetilde{U}^*U\Sigma = \widetilde{\Sigma}\widetilde{V}^*(I-D_2^{-1})V+\widetilde{U}^*(D_1^*-I)U\Sigma, \tag{7}$$

$$U^*\widetilde{U}\widetilde{\Sigma}-\Sigma V^*\widetilde{V} = U^*(I-D_1^{-*})\widetilde{U}\widetilde{\Sigma}+\Sigma V^*(D_2-I)\widetilde{V}. \tag{8}$$

The first $n$ rows of equation (7) yields

$$\widetilde{\Sigma}_1\widetilde{V}^*V-\widetilde{U}_1^*U_1\Sigma_1=\widetilde{\Sigma}_1\widetilde{V}^*(I-D_2^{-1})V+\widetilde{U}_1^*(D_1^*-I)U_1\Sigma_1. \tag{9}$$

The first $n$ rows of equation (8) yields

$$U_1^*\widetilde{U}_1\widetilde{\Sigma}_1-\Sigma_1V^*\widetilde{V}=U_1^*(I-D_1^{-*})\widetilde{U}_1\widetilde{\Sigma}_1+\Sigma_1V^*(D_2-I)\widetilde{V},$$

on taking conjugate transpose of which, one has

$$\widetilde{\Sigma}_1\widetilde{U}_1^*U_1-\widetilde{V}^*V\Sigma_1=\widetilde{\Sigma}_1\widetilde{U}_1^*(I-D_1^{-1})U_1+\widetilde{V}^*(D_2^*-I)V\Sigma_1. \tag{10}$$

Now subtracting (10) from (9) leads to

$$\begin{aligned}
&\widetilde{\Sigma}_1(\widetilde{U}_1^*U_1-\widetilde{V}^*V)+(\widetilde{U}_1^*U_1-\widetilde{V}^*V)\Sigma_1 \\
&\quad=\widetilde{\Sigma}_1\left[\widetilde{U}_1^*(I-D_1^{-1})U_1-\widetilde{V}^*(I-D_2^{-1})V\right]+\left[\widetilde{V}^*(D_2^*-I)V-\widetilde{U}_1^*(D_1^*-I)U_1\right]\Sigma_1.
\end{aligned} \tag{11}$$

To continue, we need a lemma from Li [10].

**Lemma 1** *Let $\Omega\in\mathbf{C}^{s\times s}$ and $\Gamma\in\mathbf{C}^{t\times t}$ be two Hermitian matrices, and let $E,F\in\mathbf{C}^{s\times t}$. If $\lambda(\Omega)\bigcap\lambda(\Gamma)=\emptyset$, where $\lambda(\,\cdot\,)$ is the spectrum of a matrix, then matrix equation $\Omega X-X\Gamma=\Omega E+F\Gamma$, has a unique solution, and moreover $\|X\|_{\mathrm{F}}\le\sqrt{\|E\|_{\mathrm{F}}^2+\|F\|_{\mathrm{F}}^2}\Big/\eta$, where $\eta\stackrel{\mathrm{def}}{=}\min\limits_{\omega\in\lambda(\Omega),\,\gamma\in\lambda(\Gamma)}\frac{|\omega-\gamma|}{\sqrt{|\omega|^2+|\gamma|^2}}$.*

Apply this lemma to equation (11) with $\Omega = \widetilde{\Sigma}_1$, , $= -\Sigma_1$, and

$$\begin{aligned} X &= \widetilde{U}_1^* U_1 - \widetilde{V}^* V = \widetilde{V}^*(\widetilde{V}\widetilde{U}_1^* U_1 V^* - I)V = \widetilde{V}^*(\widetilde{Q}^* Q - I)V, \\ E &= \widetilde{U}_1^*(I - D_1^{-1})U_1 - \widetilde{V}^*(I - D_2^{-1})V, \\ \widetilde{E} &= \widetilde{V}^*(D_2^* - I)V - \widetilde{U}_1^*(D_1^* - I)U_1 \end{aligned}$$

to get $\|X\|_{\rm F} = \|\widetilde{Q}^* Q - I\|_{\rm F} \le \sqrt{\|E\|_{\rm F}^2 + \|\widetilde{E}\|_{\rm F}^2}/\eta$, where $\eta = \min\limits_{1\le i,\, j\le n} \frac{\widetilde{\sigma}_i + \sigma_j}{\sqrt{\widetilde{\sigma}_i^2 + \sigma_j^2}} \ge 1$.

**Theorem 1** *Let $B$ and $\widetilde{B} = D_1^* B D_2$ be two $m\times n$ ($m \ge n$) complex matrices having full column rank and with polar decompositions (4). Then*

$$\begin{aligned} \|Q - \widetilde{Q}\|_{\rm F} &\le \sqrt{\left(\|I - D_1^{-1}\|_{\rm F} + \|I - D_2^{-1}\|_{\rm F}\right)^2 + (\|D_2 - I\|_{\rm F} + \|D_1 - I\|_{\rm F})^2}, \quad (12) \\ &\le \sqrt{2}\sqrt{\|I - D_1^{-1}\|_{\rm F}^2 + \|I - D_2^{-1}\|_{\rm F}^2 + \|D_2 - I\|_{\rm F}^2 + \|D_1 - I\|_{\rm F}^2}. \quad (13) \end{aligned}$$

*Proof:* Inequality (13) follows from (12) by the fact that $(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2)$ for $\alpha,\ \beta \ge 0$. Now we prove (12). When $m = n$, both $Q$ and $\widetilde{Q}$ are unitary. Thus $\|\widetilde{Q}^* Q - I\|_{\rm F} = \|\widetilde{Q}^*(Q - \widetilde{Q})\|_{\rm F} = \|Q - \widetilde{Q}\|_{\rm F}$. Inequality (12) is a consequence of

$$\|E\|_{\rm F} \le \|I - D_1^{-1}\|_{\rm F} + \|I - D_2^{-1}\|_{\rm F} \quad \text{and} \quad \|\widetilde{E}\|_{\rm F} \le \|D_2^* - I\|_{\rm F} + \|D_1^* - I\|_{\rm F}. \quad (14)$$

For the case $m > n$, the last $m - n$ rows of equation (8) produce that

$$U_2^* \widetilde{U}_1 \widetilde{\Sigma} = U_2^*(I - D_1^{-*})\widetilde{U}_1 \widetilde{\Sigma}_1.$$

Since $\widetilde{B}$ has full column rank, $\widetilde{\Sigma}_1$ is nonsingular. We have $U_2^* \widetilde{U}_1 = U_2^*(I - D_1^{-*})\widetilde{U}_1$. Thus

$$\|U_2^* \widetilde{U}_1\|_{\rm F} \le \|U_2^*(I - D_1^{-*})\|_{\rm F} = \|(I - D_1^{-1})U_2\|_{\rm F}. \quad (15)$$

Notice that $(U_1 V^*, U_2) = (Q, U_2)$ is unitary. Hence $U_2^* Q = 0$, and

$$\begin{aligned} \|Q - \widetilde{Q}\|_{\rm F} &= \|(Q, U_2)^*(Q - \widetilde{Q})\|_{\rm F} = \left\| \begin{pmatrix} I - Q^*\widetilde{Q} \\ -U_2^*\widetilde{Q} \end{pmatrix} \right\|_{\rm F} \\ &= \sqrt{\|I - Q^*\widetilde{Q}\|_{\rm F}^2 + \| - U_2^*\widetilde{U}_1\widetilde{V}^*\|_{\rm F}^2} \le \sqrt{\|E\|_{\rm F}^2 + \|\widetilde{E}\|_{\rm F}^2 + \|U_2^*\widetilde{U}_1\|_{\rm F}^2}. \quad (16) \end{aligned}$$

By (15), we get

$$\begin{aligned} \|E\|_{\rm F}^2 + \|U_2^*\widetilde{U}_1\|_{\rm F}^2 &\le \left(\|(I - D_1^{-1})U_1\|_{\rm F} + \|I - D_2^{-1}\|_{\rm F}\right)^2 + \|(I - D_1^{-1})U_2\|_{\rm F}^2 \\ &= \|(I - D_1^{-1})U_1\|_{\rm F}^2 + 2\|(I - D_1^{-1})U_1\|_{\rm F}\|I - D_2^{-1}\|_{\rm F} + \|I - D_2^{-1}\|_{\rm F}^2 \\ &\quad + \|(I - D_1^{-1})U_2\|_{\rm F}^2 \\ &\le \|(I - D_1^{-1})U_1\|_{\rm F}^2 + \|(I - D_1^{-1})U_2\|_{\rm F}^2 \\ &\quad + 2\|I - D_1^{-1}\|_{\rm F}\|I - D_2^{-1}\|_{\rm F} + \|I - D_2^{-1}\|_{\rm F}^2 \\ &= \|I - D_1^{-1}\|_{\rm F}^2 + 2\|I - D_1^{-1}\|_{\rm F}\|I - D_2^{-1}\|_{\rm F} + \|I - D_2^{-1}\|_{\rm F}^2 \\ &= \left(\|I - D_1^{-1}\|_{\rm F} + \|I - D_2^{-1}\|_{\rm F}\right)^2. \end{aligned}$$

Inequality (12) is a consequence of (16), (14), and the above inequalities. ∎

Now we are in the position to apply Theorem 1 to perturbations for one-side scaling. Here we consider two $n \times n$ nonsingular matrices $B = S^* G$ and $\widetilde{B} = S^* \widetilde{G}$, where $S$ is a scaling matrix and usually diagonal. (But this is not necessary to the theorem below.) The elements of $S$ can vary wildly. $G$ is nonsingular and usually better conditioned than $B$ itself. Set

$$\Delta G \stackrel{\rm def}{=} \widetilde{G} - G.$$

$\widetilde{G}$ is guaranteed nonsingular if $\|\Delta G\|_2 \|G^{-1}\|_2 < 1$ which will be assumed henceforth. Notice that

$$\widetilde{B} = S^* \widetilde{G} = S^*(G + \Delta G) = S^* G(I + G^{-1}(\Delta G)) = B(I + G^{-1}(\Delta G)).$$

So applying Theorem 1 with $D_1 = 0$ and $D_2 = I + G^{-1}(\Delta G)$ leads to

**Theorem 2** *Let $B = S^* G$ and $\widetilde{B} = S^* \widetilde{G}$ be two $n \times n$ nonsingular matrices with the polar decompositions (4). If $\|\Delta G\|_2 \|G^{-1}\|_2 < 1$ then*

$$\begin{aligned} \|Q - \widetilde{Q}\|_{\rm F} &\le \sqrt{\|G^{-1}(\Delta G)\|_{\rm F}^2 + \left\| I - (I + G^{-1}(\Delta G))^{-1} \right\|_{\rm F}^2}, \\ &\le \sqrt{1 + \frac{1}{(1 - \|G^{-1}\|_2 \|\Delta G\|_2)^2}}\ \|G^{-1}\|_2 \|\Delta G\|_{\rm F}. \end{aligned}$$

One can deal with one-side scaling from the right in the same way.

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