On exact algorithms for mapping communicating tasks onto heterogeneous systems

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Much work in progress (jointly with Kamer Kaya, Bilkent Univ., Ankara, Turkey)

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1. The problem

2. Preliminaries
   - $A^*$-search

3. $A^*$-search for task assignment

4. Experiments
Computing model: Heterogeneous processors and network.

Application model: Communicating tasks modeled using a task interaction graph, in short TIG (vertices represent tasks and edges represent intertask communications). No precedence relation among the tasks.

Objective function: Minimize the sum of the total execution and communication costs (optimize system utilization).
Formulation

Notation

\( \mathcal{P} \): The set of \( P \) processors,

\( \mathcal{T} \): The set of \( T \) tasks,

\( \{x_{tp}\}_{T \times P} \): Expected time to compute matrix (ETC); \( x_{tp} \) denotes the execution cost of task \( t \) on processor \( p \),

\( G = (\mathcal{T}, E) \): Task interaction graph; edge \( (t, u) \in E \) is associated with a communication cost multiplier \( c_{tu} \) which incurs when the tasks \( t \) and \( u \) are assigned to different processors.

\( \{d_{pq}\}_{P \times P} \): The distance between processors, i.e., if the tasks \( t \) and \( u \) are assigned to processors \( p \) and \( q \), then a communication cost of \( c_{tu} \times d_{pq} \) is incurred. Symmetric, i.e., \( d_{pq} = d_{qp} \); and \( d_{pp} = 0 \).
Find an assignment $A : \mathcal{T} \rightarrow \mathcal{P}$ that minimizes the sum of execution and communication costs:

$$
\min \left( \sum_{t=1}^{T} \sum_{p=1}^{P} a_{tp} x_{tp} + \frac{1}{2} \sum_{(t,u) \in E} \sum_{p=1}^{P} \sum_{q=1}^{P} a_{tp} a_{uq} c_{tu} d_{pq} \right) \quad \text{subject to}
$$

$$
\sum_{p=1}^{P} a_{tp} = 1, \quad t \in \mathcal{T}
$$

$$
a_{tp} \in \{0, 1\}, \quad p \in \mathcal{P}, \quad t \in \mathcal{T}
$$

Here, if task $t$ is assigned to processor $p$, then $a_{tp} = 1$ and 0 otherwise.
The general problem is **NP-complete** [references in Bokhari, IEEE TSE (1981)].

**Polynomial-time solvable instances**

- **Two processor systems** [Stone, IEEE TSE (1977)], in the time complexity of maximum-flow algorithm,

- **TIGs in tree structure** [Bokhari, IEEE TSE (1981)] on heterogeneous networks, in $O(TP^2)$ time; on homogeneous networks [Billionnet, IEEE TPDS (1994)], in $O(TP)$ time,

- **TIGs in series-parallel graph structure** [Towsley, IEEE TSE (1986)], in $O(TP^3)$ time,

- **TIGs in partial $k$-tree structure** [Fernandez-Baca, IEEE TSE (1989)], in $O(TP^{k+1})$ time.
Some recent works

Heuristics

- Minimize the completion time [Arafeh, Day, and Touzene, JSA (2007)],
- Minimize the total cost [U., Aykanat, Kaya, and Ikinci, JPDC (2006)],
- Total communication cost in heterogeneous network [Orduña, Silla, and Duato, JSA (2004)],
- Homogeneous processors and heterogeneous network [Senar, Ripoll, Cortés, and Luque, JSA (2003)],

Exact algorithms

- Homogeneous processors, heterogeneous network [Ma, Chen, and Chung, JPDC (2004)],
- Heterogeneous processors and network [Tom and Murty, Sys. Soft. (1999)],
- Heterogeneous processors, homogeneous network [Kafil and Ahmad, IEEE Concurrency (1998)].
Our aim and contributions

**Aim**

Exact algorithms for small instances (problem size is small but the search-space is huge, $P^T$),

- performance is of utmost importance,
- can be used to evaluate heuristic algorithms.

**Contributions**

- An exact algorithm using $A^*$-search,
- Use of graph theoretical concepts to reduce the search-space size,
- Use of polynomial time exact algorithms within the $A^*$-search.
A* is a best-first, graph search algorithm [Russell and Norvig, AIMA (2003)]. It finds a least cost path from a given initial node to a goal node.

Evaluation function of a node \( v \)

\[
f(v) = g(v) + h(v)
\]

- \( g(v) \) is the actual cost to reach the node \( v \) from the initial node,
- \( h(v) \) is the estimated cost to a goal node from \( v \).

The function \( h(v) \) should be an \textit{admissible} heuristic, i.e., should never overestimate the actual cost from \( v \) to a goal node.

A node with the minimum \( f \) value is expanded, i.e., all of its successors are generated and the \( f \) value for each one is computed.
**A*-search (Task assignment)**

**Search-space** is in the form of a tree

- **Initial node**: no assignment; **Goal nodes**: all tasks are assigned;
- **Intermediate node** \( v = \langle t, p \rangle \) at level \( t \): the decision of assigning task \( t \) to processor \( p \) is appended to the partial solution.

**Initialization**: \( P \) nodes

- \( \langle 1, p \rangle \) for \( 1 \leq p \leq P \), corresponding to the assignment of task 1 to the processor \( p \).
- \( g(1, p) = x_{1p} \) and \( h(1, p) \) is an estimate of the cost of assigning the remaining tasks to the processors with the information that the task 1 is assigned to the processor \( p \).

**Expanding \( \langle t, q \rangle \) — has minimum \( f \) value**

\( P \) nodes \( \langle t + 1, p \rangle \), for \( 1 \leq p \leq P \) are created, \( g \) and \( h \) values are computed and the nodes are inserted into a list using \( f \) as a key.
The functions $g(t + 1, p)$ and $h(t + 1, p)$

The actual cost, $g(t + 1, p)$, to a node

Easy to formulate using the parent node $\langle t, q \rangle$:

$$g(t + 1, p) = g(t, q) + x_{t+1,p} + \text{comm}([1, \ldots, t], t + 1)$$

Admissible heuristic $h(t + 1, p)$

Proposal: use Bokhari’s exact algorithm for the TIGs in tree structure.

Suppose we want to compute $h(t + 1, p)$ as a lower bound for the cost of assigning tasks $t + 1$ to $T$ (the tasks 1 to $t$ are assigned as in node $\langle t, p \rangle$).

Consider any spanning tree/forest of the tasks with id $t + 1$ to $T$. Run Bokhari’s algorithm on this structure; the cost is a lower bound on the cost (some adjustments are necessary).
Constructing the tree for $\langle t, p \rangle$

**Edges $(s, u)$ for $s < t \leq u$**

The task $s$ has been assigned, say to $p$. Perform a set of updates:

$$x'_{uq} = x_{uq} + c_{su}d_{pq} \quad \text{for each } q$$

such that the updated costs account for the communications involving already assigned tasks (delete/nullify those edges).

Other edges $(u, v)$ for $t \leq u, v$ kept intact; no change is necessary.

For each node of the search-space, build a tree structured problem.
Bookkeeping

Build a spanning tree/forest $S(t)$ of the vertices from $t$ to $T$, for each $t$ at the beginning of the algorithm—maximum weighted spanning tree.

During the expand operation

For each node $\langle t+1, p \rangle$ of the search-space get the tree $S(t+1)$, do the necessary transformations and solve it optimally.

This is an admissible heuristic: the assignment found by the tree algorithm is the best for the remaining tasks and the subset of communications taken into account.
We can stop when a node with an exact $h$ has the minimum key $f$.

If at a search-space node $\langle t, p \rangle$ we know that the remaining tasks forms a tree/forest, $h$ is exact.

Order the tasks

Order a set of tasks with acyclic inter-task connectivity as the last tasks.

If $F$ is such a set, we reduce the search-space size from $P^T$ to $P^{(T-F)}$.

Finding the maximum cardinality acyclic subgraph in a graph is an NP-hard problem (minimum feedback vertex set). For now, we use heuristics without any guarantee [Bafna, Berman, and Fujito, SIAM JDM (1999) has an algorithm with approximation guarantee.]
We have found the last $F$ tasks.

**Order of the other tasks**

According to their $h$ value in increasing order.

*Rationale*: During the search, $g$ tends to increase and $h$ becomes more accurate as we go deeper in the search-space.

We tried some others as well (including MaxMin), but the above one looks better.
Summary

**A* for task assignment**

1. Find a maximal acyclic set $F$ of vertices
2. Order the other vertices
3. Initialize a priority queue $Q$ with $P$ nodes $\langle 1, 1 \rangle$, $\langle 1, 2 \rangle$, $\ldots$, $\langle 1, P \rangle$ with key $f = g + h$
4. While $Q \neq \emptyset$ and $t$ of first($Q$) $\notin F$ do
5. $\langle t, p \rangle \leftarrow$ extractMin($Q$)
6. Create $P$ nodes $\langle t + 1, 1 \rangle$, $\ldots$, $\langle t + 1, P \rangle$, computing $f$, $g$, and $h$; insert into $Q$
7. End while
8. Complete the solution with the $h$ of first($Q$) $\triangleright$ as first($Q$) contains a $t \in F$

- no more than 50 Million nodes in $Q$,
- constructive heuristic to get an upper bound $U$.

$A^*$ never expands a node with $f > C^*$ where $C^*$ is the optimal assignment cost.

Not inserting the nodes with $f > U$ can reduce memory requirements (does not help to reduce time though).
Experiments

Set up

- TIGs: small sparse matrices (lacking real-life applications)
  \( T = \{59, 72, 87, 209, 307\} \) tasks.
- Communication: random integers from 1–100.
- ETCs are obtained using standard methods [Ali, Siegel, Maheswaran, Hensgen, and Ali, HCW2000],
  - ETC0: low task, low machine heterogeneity,
  - ETC1: low task, high machine heterogeneity,
  - ETC2: high task, low machine heterogeneity,
  - ETC3: high task, high machine heterogeneity.

Scaled: communication-to-computation ratio \( \rho = \{0.7, 1.0, 1.4\} \).

- \( P = \{2, 3, 4, 8\} \) processors.

Created 3 random instances for each \( T, P, \rho \) triplet.
Experiments

An alternative heuristic $h$ from the literature

Ignore all communications, and assign the tasks to their best processor [Kafil and Ahmad, IEEE Concurrency (1998)].

As in the proposed heuristic function $h$, ordering a set of independent tasks last helps reduce the search-space (our add on)—$h$ becomes exact.

Maximum independent set problem is NP-hard too.
Experiments

<table>
<thead>
<tr>
<th>$T$</th>
<th>$h=$Independent set</th>
<th></th>
<th>$h=$Tree</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P$</td>
<td></td>
<td>$P$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>59</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>E012-</td>
</tr>
<tr>
<td>72</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>E012-</td>
</tr>
<tr>
<td>87</td>
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<td>E012-</td>
<td>E012-</td>
<td>E012-</td>
</tr>
<tr>
<td>209</td>
<td>E012-</td>
<td>E012-</td>
<td>E012-</td>
<td>E-1-</td>
</tr>
<tr>
<td>307</td>
<td>E012-</td>
<td>E-1-</td>
<td>E****</td>
<td>E****</td>
</tr>
</tbody>
</table>

$\square$: all instances are solved (all types of ETC and $\rho$, 3 random instances).

$E012-$: all instances with ETC type 0, 1, and 2 are solved. For ETC of type 3 some or all instances needed more than 50 million nodes, hence exited without solving (for some $\rho$).

*: were still running when I left the office (may or may not obtain solution).
# Experiments: Some comparisons with the independent set heuristic

<table>
<thead>
<tr>
<th>Problem</th>
<th>Metric</th>
<th>( h = \text{Independent set} )</th>
<th>( h = \text{Tree} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T = 307 ) ( P = 3 ) ( \rho = 1.4 ) ETC2</td>
<td>Search-space size</td>
<td>( 3^{307-79} )</td>
<td>( 3^{307-147} )</td>
</tr>
<tr>
<td></td>
<td>Opened nodes</td>
<td>22,917,457</td>
<td>22,190</td>
</tr>
<tr>
<td></td>
<td>Time (s.)</td>
<td>4,858</td>
<td>4</td>
</tr>
<tr>
<td>( T = 59 ) ( P = 8 ) ( \rho = 1.0 ) ETC3</td>
<td>Search-space size</td>
<td>( 8^{59-22} )</td>
<td>( 8^{59-38} )</td>
</tr>
<tr>
<td></td>
<td>Opened nodes</td>
<td>46,246,756</td>
<td>1849</td>
</tr>
<tr>
<td></td>
<td>Time (s.)</td>
<td>10,617</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(Runs are on a machine with 64 AMD Dual 250 Opteron.)

**Reminders**

- ETC2: high task, low machine heterogeneity.
- ETC3: high task and machine heterogeneity.
- \( \rho \): communication-to-computation ratio.
Conclusion and future work

Aiming at solving problems of size $T \geq 500$ and $P \geq 4$ (hopefully on real life applications).

Plans

- implement memory efficient variants of $A^*$,
- use constructive heuristics to find upper bounds at different levels to save some memory (like branch-and-bound algorithm),
- implement the exact algorithms for other special cases of the problem (e.g., series-parallel graphs) to have a better heuristic function $h$.
- the quadratic assignment problem bears similarities. Investigate the applicability there (the largest instance in QAPLIB has a size of $n = 250$, most are smaller than 100, not all have been solved optimally—this time search-space is usually of size $n^n$).
Further information

Thank you for your attention.

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